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Symmetry in a constrained Hamiltonian system with singular higher-order Lagrangian

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Abstract. For a canonical formalism with a higher-order derivative, the corresponding generalized first Noether theorem (GFNT) for a constrained Hamiltonian system and the generalized Noether identities (GNI) for a system with a non-invariant action integral are derived, which may be useful to analyse the Dirac constraint for such a system. Using the GFNT another example is given in which Dirac's conjecture fails; using the GNI the strong and weak conservation laws are deduced and it is pointed out that for certain variant systems there is also a Dirac constraint. Suppose that there are only first-class constraints (FCC) in a system, then an algorithm for the construction of a gauge generator is developed, once the Hamiltonian and the FCC of the system with a higher-order Lagrangian are given.

1. Introduction

The connection between continuous symmetry and conservation laws is usually referred to as Noether's theorem. In previous papers the generalization of the first Noether theorem for constrained and non-conservative systems (Li 1981, 1984, 1985, Li and Li 1990) and the generalization of Noether identities for variant systems (Li 1987, 1988) were given. In these papers, all considerations are based on examination of the Lagrangian in configuration space and the corresponding transformations expressed in terms of Lagrange's variables. For a system with regular Lagrangian and finite degrees of freedom, the invariance under the continuous transformation in terms of Hamilton's variables was discussed by Djukic (1974). The system with a singular Lagrangian is subject to some inherent phase space constraint (Dirac 1964, Sundermeyer 1982), the generalization of the Noether theorem to a system with ordinary-singular Lagrangian in terms of canonical variables was discussed by the author (Li and Li 1991). Here the symmetry properties in a constrained Hamiltonian system with a singular higher-order Lagrangian are further investigated.

Dynamical systems described in terms of higher derivatives have been investigated for a long time (Leon and Rodrigues 1985) in connection with non-local field theory (Pais and Uhlenbeck 1950), relativistic dynamics of particles (Ellis 1975, Jaen *et al* 1986, Nesterenko 1989), gravity theory (Utiyama and DeWitt 1962, Stelle 1977, Fradkin and Tseytlin 1982, Szczyrba 1987), modified KdV equations (Kentwell 1988), supersymmetry (Kersten 1988), string models (Battle *et al* 1987, Nesterenko and Nguyen 1988) and so on (Galvão and Lemos 1988, Saito *et al* 1989, Hebda 1990). To analyse the symmetry properties of a singular Lagrangian with higher derivatives is thus necessary.

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In this paper, the generalized first Noether theorem (GFNT) for a constrained Hamiltonian system and the generalized Noether identities (GNI) for a variant system with a second-order Lagrangian are derived in a canonical formalism, and some applications of these theorems to the analysis of the Dirac constraint are given. From the GNI the strong and weak conservation laws can be deduced. Using the GNI it is shown that for certain variant systems there is also a Dirac constraint. Combining the GNI and Dirac constraint conditions gives rise to more relationships among some of the variables. As is well known, the Lagrangian multipliers connecting with first-class constraints (FCC) represent the functional arbitrariness in the theory. Along the trajectory of motion the GNI can give us some additional information about Lagrange multiplers connecting with FCC. The GFNT and GNI can give us another possibility to discuss Dirac's conjecture. When the solutions of generalized Killing's equations are found, then the conserved quantities of the form (20) automatically exist. Whether the conservation laws (20) derived from H_E via canonical formalism are exactly equivalent to the results arising from Lagrange's formalism via the classical Noether theorem is considered. Another example is given in which Dirac's conjecture fails. Moreover, one can examine the GNI which give us the consistency or inconsistency results along the trajectory of the constrained system arising from H_E for an admissible Lagrangian; if one obtains an inconsistency result, then Dirac's conjecture fails in that problem. Suppose that there are only FCC in a system or the FCC are completely separated from the series of second-class constraints (SCC), then an algorithm which gives the canonical gauge generators for a constrained Hamiltonian system with a singular second-order Lagrangian is developed. An example is given: application of these results to a model of field theories for which the gauge transformation has been constructed and a conservative current found for the field coupling with an external source; this conservation law is valid whether Dirac's conjecture holds true or not.

2. GFNT in a canonical formalism for a constrained Hamiltonian system

The GFNT will be given in a canonical formalism for a constrained Hamiltonian system with a singular second-order Lagrangian. For the sake of simplicity one usually considers a system with finite degrees of freedom exhibiting the essential problem by invariant theories and extension to field theories, and the singular Nth-order Lagrangian is straightforward. Consider a system described by a singular Lagrangian $L=L(t, q, \dot{q}, \ddot{q})$ $(q=[q^1, q^2, \ldots, q^N])$. The Ostrogradski transformation introduces canonical momenta (Nesterenko 1989)

$$p_i^{(1)}=\frac{\partial L}{\partial q_{(2)}^i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_{(2)}^i} \tag{1}$$

$$p_i^{(2)}=\frac{\partial L}{\partial \dot{q}_{(2)}^i} \tag{2}$$

where $q_{(1)}^i=q^i$, $q_{(2)}^i=\dot{q}^i$, and using these relations one can go over from the Lagrangian description to the Hamiltonian description. The canonical Hamiltonian is defined by

$$H=\dot{q}_{(1)}^i p_i^{(1)}+\dot{q}_{(2)}^i p_i^{(2)}-L \tag{3}$$

which may be formed by eliminating only the $\dot{q}_{(2)}^i$. The summation is taken over repeated indices. The Hamiltonian of the system depends only on the canonical variables both for regular and singular Lagrangians. From the variation of the Hamiltonian and Euler-Lagrange equations one can get (Nesterenko 1989)

$$\frac{\delta I}{\delta p_i^{(\alpha)}}\,\delta p_i^{(\alpha)}+\frac{\delta I}{\delta q_{(\alpha)}^i}\,\delta q_{(\alpha)}^i=0 \tag{4}$$

where

$$\frac{\delta I}{\delta p_i^{(\alpha)}}=\dot{q}_{(\alpha)}^i-\frac{\partial H}{\partial p_i^{(\alpha)}} \qquad \frac{\delta I}{\delta q_{(\alpha)}^i}=-\left(\dot{p}_i^{(\alpha)}+\frac{\partial H}{\partial q_{(\alpha)}^i}\right). \tag{5}$$

For a regular Lagrangian the canonical variables $q_{(\alpha)}^i$, $p_i^{(\alpha)}$ are independent, but for a singular Lagrangian the extended Hessian matrix

$$H_{ij}=\frac{\partial^2 L}{\partial \dot{q}_{(2)}^i\partial \dot{q}_{(2)}^j} \tag{6}$$

is degenerate and one supposes its rank to be $N-R$. Since H_{ij} is degenerate, (2) is not solvable for all $\dot{q}_{(2)}^i$, but, we have R constraints,

$$\phi_a^0(q_{(\alpha)},p^{(\alpha)})=0 \qquad (a=1,2,\ldots,R,\ \alpha=1,2) \tag{7}$$

which are derived from (2) and called the primary constraints (PC). From (7) one has

$$\frac{\partial \phi_a^0}{\partial q_{(\alpha)}^i}\,\delta q_{(\alpha)}^i+\frac{\partial \phi_a^0}{\partial p_i^{(\alpha)}}\,\delta p_i^{(\alpha)}=0. \tag{8}$$

Introducing the Lagrangian multipliers $\lambda^a(t)$ and combining the expressions (4) and (8) one obtains the canonical equations for a constrained Hamiltonian system with a singular second-order Lagrangian:

$$\frac{\delta I}{\delta q_{(\alpha)}^i}=\lambda^a\frac{\partial \phi_a^0}{\partial q_{(\alpha)}^i} \qquad \frac{\delta I}{\delta p_i^{(\alpha)}}=\lambda^a\frac{\partial \phi_a^0}{\partial p_i^{(\alpha)}}. \tag{9}$$

Using the Poisson bracket

$$\{u,v\}=\frac{\partial u}{\partial q_{(\alpha)}^i}\frac{\partial v}{\partial p_i^{(\alpha)}}-\frac{\partial u}{\partial p_i^{(\alpha)}}\frac{\partial v}{\partial q_{(\alpha)}^i} \tag{10}$$

(9) can be written as

$$\dot{q}_{(\alpha)}^i=\{q_{(\alpha)}^i,H_{\mathrm{T}}\} \qquad p_i^{(\alpha)}=\{p_i^{(\alpha)},H_{\mathrm{T}}\} \tag{11}$$

where H_{T} is a total Hamiltonian, $H_{\mathrm{T}}=H+\lambda^a\phi_a^0$.

Suppose that for a system it is possible to construct a Lagrangian L_p in Hamilton's form and that the corresponding action integral is

$$I=\int_{t_1}^{t_2}L_p\,\mathrm{d}t=\int_{t_1}^{t_2}(\dot{q}_{(\alpha)}^i p_i^{(\alpha)}-H(q_{(\alpha)},p^{(\alpha)}))\,\mathrm{d}t. \tag{12}$$

Let us consider the transformation properties of the system under the continuous group with the infinitesimal transformation given by

$$\begin{aligned}
&t'=t+\Delta t=t+\varepsilon_\sigma\tau^\sigma(t,q_{(\alpha)},p^{(\alpha)})\\
&q_{(\alpha)}^{i'}(t')=q_{(\alpha)}^i(t)+\Delta q_{(\alpha)}^i(t)=q_{(\alpha)}^i(t)+\varepsilon_\sigma\xi_{(\alpha)}^{i\sigma}(t,q_{(\alpha)},p^{(\alpha)})\\
&p_i^{(\alpha)'}(t')=p_i^{(\alpha)}(t)+\Delta p_i^{(\alpha)}(t)=p_i^{(\alpha)}(t)+\varepsilon_\sigma\eta_i^{\sigma(\alpha)}(t,q_{(\alpha)},p^{(\alpha)})
\end{aligned} \tag{13}$$

where ε_σ $(\sigma=1,2,\ldots,r)$ are parameters. Suppose the change of the Lagrangian is $\delta L_p=D(\varepsilon_\sigma\Omega^\sigma)$ under the transformation (13), where $D=\mathrm{d}/\mathrm{d}t$ and $\Omega^\sigma=\Omega^\sigma(t,q_{(\alpha)},p^{(\alpha)})$, i.e. the Lagrangian is invariant up to an exact differential term under the transformation (13), then from (12) one has

$$\frac{\delta I}{\delta p_i^{(\alpha)}}\,\delta p_i^{(\alpha)}+\frac{\delta I}{\delta q^i_{(\alpha)}}\,\delta q^i_{(\alpha)}+D(p_i^{(\alpha)}\delta q^i_{(\alpha)}+L_p\Delta t)=D(\varepsilon_\sigma\Omega^\sigma) \tag{14}$$

where

$$\delta p_i^{(\alpha)}=\Delta p_i^{(\alpha)}-\dot p_i^{(\alpha)}\Delta t \qquad\qquad \delta q^i_{(\alpha)}=\Delta q^i_{(\alpha)}-\dot q^i_{(\alpha)}\Delta t. \tag{15}$$

Under the transformation (13), the change of the ϕ_a^0 are given by $\delta\phi_a^0=\varepsilon_\sigma K_a^\sigma$, then one has

$$\delta\phi_a^0=\frac{\partial\phi_a^0}{\partial q^i_{(\alpha)}}\,\delta q^i_{(\alpha)}+\frac{\partial\phi_a^0}{\partial p_i^{(\alpha)}}\,\delta p_i^{(\alpha)}=\varepsilon_\sigma F_a^\sigma \tag{16}$$

where

$$F_a^\sigma=K_a^\sigma-\frac{\partial\phi_a^0}{\partial q^i_{(\alpha)}}\,\dot q^i_{(\alpha)}\tau^\sigma-\frac{\partial\phi_a^0}{\partial p_i^{(\alpha)}}\,\dot p_i^{(\alpha)}\tau^\sigma. \tag{17}$$

Using a set of the Lagrange multipliers $\lambda^a(t)$ and combining the expressions (14) and (16), from (9) one obtains

$$D[p_i^{(\alpha)}(\xi^{i\sigma}-\dot q^i_{(\alpha)}\tau^\sigma)+L_p\tau^\sigma-\Omega^\sigma]=\lambda^a F_a^\sigma. \tag{18}$$

Therefore, we have the following GFNT in canonical formalism. If, under the transformation (13), the Lagrangian L_p is invariant up to an exact differential term such that the constraint conditions satisfy $F_a^\sigma=0$, or

$$\frac{\partial\phi_a^0}{\partial q^j_{(\alpha)}}(\xi^{i\sigma}_{(\alpha)}-\dot q^i_{(\alpha)}\tau^\sigma)+\frac{\partial\phi_a^0}{\partial p_i^{(\alpha)}}(\eta_i^{\sigma(\alpha)}-\dot p_i^{(\alpha)}\tau^\sigma)=0 \tag{19}$$

then the expressions

$$p_i^{(\alpha)}\xi^{i\sigma}_{(\alpha)}-H\tau^\sigma-\Omega^\sigma=\text{const} \qquad\qquad (\sigma=1,2,\ldots,r) \tag{20}$$

are constants of the motion. This theorem is a generalization of the previous result (Li and Li 1991).

If $\Delta t=0$ in the transformation (13), condition (19) implies that the constraint conditions are invariant under the transformation (13); if $\Delta t\neq 0$ in the transformation (13), the condition (19) implies that the constraint conditions are invariant under the simultaneous variations $\delta q^i_{(\alpha)}$ and $\delta p_i^{(\alpha)}$ determined by (13).

If by assumption τ^σ, $\xi^{i\sigma}_{(\alpha)}$ and $\eta_i^{\sigma(\alpha)}$ do not depend on $\dot q^i_{(2)}$ and $\dot p_i^{(\alpha)}$ $(\alpha=1,2)$, substituting (13) in the necessary condition (14) which must be satisfied for a Lagrangian L_p to be invariant up to an exact differential term, then the condition (14) leads to the systems of partial differential equations of unknown variables τ^σ, $\xi^{i\sigma}_{(\alpha)}$ and $\eta_i^{\sigma(\alpha)}$ obtained by equating terms in corresponding degrees of $\dot q^i_{(2)}$ and $\dot p_i^{(\alpha)}$ on the left- and right-hand sides of (14) (e.g. see Djukic 1974):

$$p_i^{(\alpha)}\frac{\partial\xi^{i\sigma}_{(\alpha)}}{\partial p_j^{(1)}}-H\frac{\partial\tau^\sigma}{\partial p_j^{(1)}}=\frac{\partial\Omega^\sigma}{\partial p_j^{(1)}} \tag{21a}$$

$$p_i^{(\alpha)}\frac{\partial\xi^{i\sigma}_{(\alpha)}}{\partial p_j^{(2)}}-H\frac{\partial\tau^\sigma}{\partial p_j^{(2)}}=\frac{\partial\Omega^\sigma}{\partial p_j^{(2)}} \tag{21b}$$

$$\eta_j^{\sigma(2)}+p_i^{(\alpha)}\frac{\partial\xi^{i\sigma}_{(\alpha)}}{\partial q^j_{(2)}}-H\frac{\partial\tau^\sigma}{\partial q^j_{(2)}}=\frac{\partial\Omega^\sigma}{\partial q^j_{(2)}} \tag{21c}$$

$$\eta_i^{\sigma(1)} q_{(2)}^i - \eta_i^{\sigma(\alpha)} \frac{\partial H}{\partial p_i^{(\alpha)}} + p_i^{(\alpha)} \left(\frac{\partial \xi_{(\alpha)}^{i\sigma}}{\partial t} + \frac{\partial \xi_{(\alpha)}^{i\sigma}}{\partial q_{(1)}^j} q_{(2)}^j \right) - \xi_{(1)}^{i\sigma} \left(\frac{\partial H}{\partial q_{(1)}^i} + \frac{\partial H}{\partial q_{(2)}^i} \right)$$
$$- H \left(\frac{\partial \tau^\sigma}{\partial t} + \frac{\partial \tau^\sigma}{\partial q_{(1)}^i} q_{(2)}^i \right) - \frac{\partial H}{\partial t} \tau^\sigma$$
$$= \frac{\partial \Omega^\sigma}{\partial t} + \frac{\partial \Omega^\sigma}{\partial q_{(1)}^i} q_{(2)}^i \tag{21d}$$

these partial differential equations (21) and condition (19), which are linearly related to the unknown functions τ^σ, $\xi_{(\alpha)}^{i\sigma}$ and $\eta_i^{\sigma(\alpha)}(i=1,2,\ldots,N,\ \alpha=1,2,\ \sigma=1,2,\ldots,r)$, are called generalized Killing's equations for a constrained Hamiltonian system with a singular second-order Lagrangian. When the generalized Killing's equations, where the functions H, ϕ_a^0 and Ω^σ are defined, admit a solution in τ^σ, $\xi_{(\alpha)}^{i\sigma}$ and $\eta_i^{\sigma(\alpha)}$, then the conserved quantities of the form (20) automatically exist for this system.

3. Dirac's conjecture

Dirac (1964) in his work on generalized canonical formalism conjectured that all secondary first-class constraints (SFCC) are independent generators of the gauge transformation which generates equivalence transformations among physical states. If this conjecture holds true, then the dynamics of a constrained Hamiltonian system should be correctly described by the equations of motion arising from the extended Hamiltonian $H_E = H_T + \mu^a \chi_a$, where χ_a are SFCC and μ^a are Lagrange multipliers. There have been some objections to Dirac's conjecture (Sugano and Kamo 1982, Appleby 1982, Castellani 1982, Sugano and Kimura 1983, Costa *et al* 1985, Grácia and Pons 1988), and some counter examples have been given (Cawley 1979, Frenkel 1980). All these objections are based on the straightforward observation that the equations of motion derived from H_E are not strictly equivalent to the corresponding Lagrange equations. Here, this problem will be discussed starting from another point of view.

In analogy to Dirac's generalized Hamiltonian dynamics, for the system with a singular higher-order Lagrangian, from the stationary of the PC, one can define successively the secondary constraints according to the Dirac–Bergmann algorithm

$$\phi_a^k = \{\phi_a^{k-1}, H_T\}. \tag{22}$$

This algorithm is continued until ϕ_a^m satisfies

$$\phi_a^{m+1} = \{\phi_a^m, H_T\} = c_{ak}^b \phi_b^k \qquad (k \leqslant m). \tag{23}$$

All the constraints ϕs are classified into two classes. A ϕ_a is defined to be first class if $\{\phi_a, \phi_b\} = 0 \ (\mathrm{mod}\ \phi_c)$ for all ϕs, otherwise it is second class. Similarly, there is also a problem about Dirac's conjecture for a singular higher-order Lagrangian.

The GFNT in canonical formalism gives us another possibility to examine Dirac's conjecture. Let us consider whether the conservation laws derived from H_E via GFNT in canonical formalism are equivalent to the results arising from Lagrange's formalism via the classical Noether theorem. Now we present another example with the Lagrangian

$$L = \dot{x}^n(\dot{z}_1^2 + \dot{z}_2^2) + \tfrac{1}{2} y (z_1^2 + z_2^2) \qquad (n \geqslant 1) \tag{24}$$

the previous example (Li and Li 1991) is a special case for $n=1$ (see also Frenkel 1980). There is only a primary constraint

$$\phi^0 = p_y = 0. \tag{25}$$

The generalized velocity $\dot{x}$ can be represented by

$$\dot{x}=\left(\frac{n}{4p_x}\right)^{1/(n+1)}(p_{z_1}^2+p_{z_2}^2)^{1/(n+1)}. \tag{26}$$

The total Hamiltonian is

$$H_{\mathrm{T}}=\frac{n+1}{4}\left(\frac{4}{n}\right)^{n/(n+1)}p_x^{n/(n+1)}(p_{z_1}^2+p_{z_2}^2)^{1/(n+1)}-\tfrac{1}{2}y(z_1^2+z_2^2)+\lambda p_y. \tag{27}$$

The stationary condition of constraint gives us the following secondary constraints as long as $x\neq$ const:

$$\chi_1=z_1^2 \tag{28a}$$

$$\chi_2=z_2^2 \tag{28b}$$

$$\chi_3=z_1p_{z_1} \tag{28c}$$

$$\chi_4=z_2p_{z_2} \tag{28d}$$

$$\chi_5=p_x. \tag{28e}$$

All these constraints are FCC. The extended Hamiltonian is

$$H_{\mathrm{E}}=H_{\mathrm{T}}+\mu^a\chi_a \tag{29}$$

where μ^a $(a=1,2,\ldots,5)$ are Lagrangian multipliers. The Lagrangian L_p and ϕ^0 are invariant under the rotation in the (z_1,z_2) plane. From the H_{T} via the GFNT in canonical formalism one can obtain angular momentum conservation which can also be yielded by Lagrange's variables via the classical Noether theorem. But if the SFCC in the Hamiltonian are taken into account one cannot obtain this result from the extended Hamiltonian H_{E}. Dirac's conjecture fails in this example, which differs from other counter examples in that we do not write constraints in a linearized form as Cawley and others do. Most recently, it has been shown by Qi (1990) that for the examples of Cawley and others Dirac's conjecture holds true.

4. GNI in canonical formalism for a constrained Hamiltonian system

As is well known, in the massive Yang–Mills theories the Lagrangian in general is not invariant under gauge transformation; the gauge-invariant Lagrangian of Fermi and gauge fields is not invariant under the chirality transformation of the Fermi field; the invariant Lagrangian under the BRS transformation is not invariant under the gauge transformation alone, etc. Therefore the discussion of the transformation properties for variant system is necessary (Li 1987). Now let us consider the transformation depending on arbitrary functions $\varepsilon_\sigma(t)$ $(\sigma=1,2,\ldots,r)$ and their derivatives up to some fixed order. Such infinitesimal transformations in canonical variables are given by

$$\begin{aligned}
t'&=t+R^\sigma\varepsilon_\sigma=t+a_k^\sigma D^k\varepsilon_\sigma(t)\\
q_{(\alpha)}^{i'}(t')&=q_{(\alpha)}^i(t)+S_{(\alpha)}^{i\sigma}\varepsilon_\sigma=q_{(\alpha)}^i(t)+b_{l(\alpha)}^{i\sigma}D^l\varepsilon_\sigma(t)\\
p_i^{(\alpha)'}(t')&=p_i^{(\alpha)}(t)+T_i^{\sigma(\alpha)}\varepsilon_\sigma=p_i^{(\alpha)}(t)+c_{im}^{\sigma(\alpha)}D^m\varepsilon_\sigma(t)
\end{aligned} \tag{30}$$

where a_k^σ, $b_{l(\alpha)}^{i\sigma}$ and $c_{im}^{\sigma(\alpha)}$ are the functions of t, $q_{(\alpha)}^i$ and $p_i^{(\alpha)}$. In the gauge field theories, according to the gauge transformation of field variables, one can in general define the transformation of canonical momenta. The transformation properties of the system with respect to this transformation leads to the GNI in canonical formalism. In quantum theory, the GNI correspond to the Ward–Takahashi identities.

Under the transformation (30) suppose that the change of canonical action (12) is given by

$$\delta I=\int_{t_1}^{t_2}(D(\Omega^\sigma\varepsilon_\sigma)+U^\sigma\varepsilon_\sigma)\,\mathrm{d}t \tag{31}$$

where U^σ and Ω^σ are linear differential operators,

$$U^\sigma=u_n^\sigma D^n \qquad \Omega^\sigma=v_j^\sigma D^j \tag{32}$$

and the u's and v's are functions of t, $q^i_{(\alpha)}$ and $p_i^{(\alpha)}$. From expression (12) one has

$$\begin{aligned}\int_{t_1}^{t_2}\left(\frac{\delta I}{\delta p_i^{(\alpha)}}(T_i^{\sigma(\alpha)}-\dot p_i^{(\alpha)}R^\sigma)+\frac{\delta I}{\delta q^i_{(\alpha)}}(S^{i\sigma}_{(\alpha)}-\dot q^i_{(\alpha)}R^\sigma)\right)\varepsilon_\sigma\,\mathrm{d}t\\ +[p_i^{(\alpha)}(S^{i\sigma}_\alpha-\dot q^i_{(\alpha)}R^\sigma)+L_pR^\sigma-\Omega^\sigma]\varepsilon_\sigma\Big|_{t_1}^{t_2}\\ =\int_{t_1}^{t_2}U^\sigma\varepsilon_\sigma\,\mathrm{d}t.\end{aligned} \tag{33}$$

Since $\varepsilon_\sigma(t)$ are arbitrary, one may choose $\varepsilon_\sigma(t)$ and their derivatives up to a required order such that the boundary term in (33) vanishes, and repeat the integration by parts of the remaining terms of this identity. Again considering the arbitrariness of the $\varepsilon_\sigma(t)$, one can force the boundary term to vanish, after which one can apply the fundamental lemma of the calculus of variations to conclude that

$$\tilde T_i^{\sigma(\alpha)}\left(\frac{\delta I}{\delta p_i^{(\alpha)}}\right)-\tilde R^\sigma\left(\dot p_i^{(\alpha)}\frac{\delta I}{\delta p_i^{(\alpha)}}\right)+\tilde S^{i\sigma}_{(\alpha)}\left(\frac{\delta I}{\delta q^i_{(\alpha)}}\right)-\tilde R^\sigma\left(\dot q^i_{(\alpha)}\frac{\delta I}{\delta q^i_{(\alpha)}}\right)=\tilde U^\sigma(1)\qquad(\sigma=1,2,\ldots,r) \tag{34}$$

where $\tilde R^\sigma$, $\tilde S^{i\sigma}_{(\alpha)}$, $\tilde T_i^{\sigma(\alpha)}$ and $\tilde U^\sigma$ are the adjoint operators with respect to R^σ, $S^{i\sigma}_{(\alpha)}$, $T_i^{\sigma(\alpha)}$ and U^σ respectively, defined by

$$\int_a^b fR^\sigma g\,\mathrm{d}t=\int_a^b g\tilde R^\sigma f\,\mathrm{d}t+[\cdot]_a^b \tag{35}$$

where f, g are smoothed functions defined on $[a,b]$ and $[\cdot]_a^b$ represent the boundary terms, and similar expressions hold for $S^{i\sigma}_{(\alpha)}$, $\tilde S^{i\sigma}_{(\alpha)}$; $T_i^{\sigma(\alpha)}$, $\tilde T_i^{\sigma(\alpha)}$ and U^σ, $\tilde U^\sigma$. In (34), $\tilde U^\sigma(1)$ indicates the adjoint operator applied to unity. The expressions (34) are called GNI in canonical variables of a system whose action integral is variant under the transformation (30). This is a generalization of previous work (Li and Li 1991). We combine the constraint conditions for the constrained system and the GNI, which may give rise to more relationships among some of the variables. Sometimes these can tell us at what stage the Dirac–Bergmann algorithm will terminate.

According to the GNI (34), for certain cases, one can obtain the strong conservation laws or exact differential identities which are valid whether the equations of motion are satisfied or not.

Suppose in transformation (30), $R^\sigma=a_0^\sigma$, $S^{i\sigma}_{(\alpha)}=b^{i\sigma}_{0(\alpha)}+b^{i\sigma}_{1(\alpha)}D$, $T_i^{\sigma(\alpha)}=c_{i0}^{\sigma(\alpha)}+c_{i1}^{\sigma(\alpha)}D$ and $U^\sigma=u_0^\sigma+u_1^\sigma D+u_2^\sigma D^2$, where a_0^σ, $b^{i\sigma}_{0(\alpha)}$, $b^{i\sigma}_{1(\alpha)}$, $c_{i0}^{\sigma(\alpha)}$, $c_{i1}^{\sigma(\alpha)}$, u_0^σ, u_1^σ

and u_2^σ are functions of t, $q^i_{(\alpha)}$ and $p_i^{(\alpha)}$. Multipling the GNI (34) by $\varepsilon_\sigma(t)$ and substracting the result from the basic identity

$$\left(\frac{\delta I}{\delta p_i^{(\alpha)}}(T_i^{\sigma(\alpha)}-\dot{p}_i^{(\alpha)}a_0^\sigma)+\frac{\delta I}{\delta q^i_{(\alpha)}}(S^{i\sigma}_{(\alpha)}-\dot{q}^i_{(\alpha)}a_0^\sigma)\right)\varepsilon_\sigma+D\{[p_i^{(\alpha)}(S^{i\sigma}_{(\alpha)}-\dot{q}^i_{(\alpha)}a_0^\sigma)+L_p a_0^\sigma]\varepsilon_\sigma\}$$
$$=D(\Omega^\sigma\varepsilon_\sigma)+U^\sigma\varepsilon_\sigma \tag{36}$$

one obtains the exact differential identity

$$D\left[\left(b^{i\sigma}_{1(\alpha)}\frac{\delta I}{\delta q^i_{(\alpha)}}+c_{i1}^{\sigma(\alpha)}\frac{\delta I}{\delta p_i^{(\alpha)}}+p_i^{(\alpha)}(b^{i\sigma}_{0(\alpha)}+b^{i\sigma}_{1(\alpha)}D)-Ha_0^\sigma-\Omega^\sigma-u_1^\sigma\right)\varepsilon_\sigma\right.$$
$$\left.+\varepsilon_\sigma Du_2^\sigma-u_2^\sigma D\varepsilon_\sigma\right]=0. \tag{37}$$

If the transformation group has a subgroup and $\varepsilon_0(t)=\varepsilon_p^0\Lambda_\sigma^\rho(t)$, where ε_p^0 are parameters of the Lie group, one can get the weak conservation laws along the trajectory of the motion

$$\left(b^{i\sigma}_{1(\alpha)}\lambda^a\frac{\partial\phi_a^0}{\partial q^i_{(\alpha)}}+c_i^{\sigma(\alpha)}\lambda^a\frac{\partial\phi_a^0}{\partial p_i^{(\alpha)}}+p_i^{(\alpha)}(b^{i\sigma}_{0(\alpha)}+b^{i\sigma}_{1(\alpha)}D)-Ha_0^\sigma-\Omega^\sigma-u_1^\sigma\right)\Lambda_\sigma^\rho$$
$$+\Lambda_\sigma^\rho Du_2^\sigma-u_2^\sigma D\Lambda_\sigma^\rho=\text{const}\qquad(\rho=1,2,\ldots,r). \tag{38}$$

Using the GNI (34) one can also discuss Dirac's conjecture, if this conjecture holds true, along the trajectory of motion arising from H_E and the GNI (34) becomes

$$\tilde{T}_i^{\sigma(\alpha)}\left(\lambda^a\frac{\partial\phi_a}{\partial p_i^{(\alpha)}}\right)-\tilde{R}^\sigma\left(\dot{p}_i^{(\alpha)}\lambda^a\frac{\partial\phi_a}{\partial p_i^{(\alpha)}}\right)+\tilde{S}^{i\sigma}_{(\alpha)}\left(\lambda^a\frac{\partial\phi_a}{\partial q^i_{(\alpha)}}\right)-\tilde{R}^\sigma\left(\dot{q}^i_{(\alpha)}\lambda^a\frac{\partial\phi_a}{\partial q^i_{(\alpha)}}\right)$$
$$=\tilde{U}^\sigma(1)\qquad(\sigma=1,2,\ldots,r). \tag{39}$$

All the FCC are taken into account in the set of ϕ_a. If (39) gives us inconsistent results for admissible Lagrangians, then Dirac's conjecture regarding the SFCC may be invalid in this circumstance.

In theories with the SCC, all the Lagrange multipliers connecting with the SCC are determined by the Hamiltonian and SCC themselves, but in theories with the FCC, the Lagrangian multipliers connecting with the FCC are not determined by the equations of motion, and the undetermined multipliers represent the whole of the functional arbitrariness in the solution of the Hamiltonian equations of motion (Sundermeyer 1982). If Dirac' conjecture holds true in a problem, along the trajectory of motion, the expression (39) may become a trivial equality or sometimes perhaps give us more relationships for these Lagrange multipliers connecting with the FCC. Therefore, the application of the GFNT and GNI in canonical formalism enables us to obtain some additional information about the Dirac constraint and corresponding Lagrange multipliers.

5. The variance and Dirac constraint

As is well known, a gauge-invariant system in Lagrangian formalism has a Dirac constraint (Sundermeyer 1982). Using the GNI (34) in canonical formalism it can further be shown that for certain variant systems are also constrained Hamiltonian systems.

Consider a system, under the local transformation

$$
\begin{aligned}
&t'=t\\
&q^{i'}_{(\alpha)}(t')=q^{i}_{(\alpha)}(t)+(b^{i\sigma}_{0(\alpha)}+b^{i\sigma}_{1(\alpha)}D)\varepsilon_\sigma(t)\\
&p^{(\alpha)'}_{i}(t')=p^{(\alpha)}_{i}(t)+(c^{\sigma(\alpha)}_{i0}+c^{\sigma(\alpha)}_{i1}D)\varepsilon_\sigma(t)
\end{aligned}
\tag{40}
$$

and suppose the change of canonical action integral is given by (31), where

$$
U^\sigma=u^\sigma_0(t,q^i_{(\alpha)},p^{(\alpha)}_i)+u^\sigma_1(t,q^i_{(\alpha)},p^{(\alpha)}_i).
\tag{41}
$$

The corresponding transformation properties of the massive Yang–Mills field theories belong to this category, when mass terms are introduced in the Lagrangian. In this case the GNI (34) become

$$
c^{\sigma(\alpha)}_{0i}\left(\dot q^i_{(\alpha)}-\frac{\partial H}{\partial p^{(\alpha)}_i}\right)-b^{i\sigma}_{0(\alpha)}\left(\dot p^{(\alpha)}_i+\frac{\partial H}{\partial q^i_{(\alpha)}}\right)-D\left[c^{\sigma(\alpha)}_{i1}\left(\dot q^i_{(\alpha)}-\frac{\partial H}{\partial p^{(\alpha)}_i}\right)\right]
$$

$$
+D\left[b^{i\sigma}_{1(\alpha)}\left(\dot p^{(\alpha)}_i+\frac{\partial H}{\partial q^i_{(\alpha)}}\right)\right]=u^\sigma_0-Du^\sigma_1.
\tag{42}
$$

We remember the Ostrogradski transformation for $p^{(1)}_i$ and

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot q^i}=\frac{\partial^2 L}{\partial \ddot q^i\,\partial t}+\frac{\partial^2 L}{\partial \ddot q^i\,\partial q^j}\dot q_j+\frac{\partial^2 L}{\partial \ddot q^i\,\partial \dot q^j}\ddot q^j+\frac{\partial^2 L}{\partial \ddot q^i\,\partial \ddot q^j}\dddot q^j.
\tag{43}
$$

Substituting (1) and (43) into the identities (42), the highest-order derivative of q_i must occur in the terms $D(b^{i\sigma}_{1(1)}\dot p^{\omega}_i)$, i.e. in the terms $D\{b^{i\sigma}_{1(1)}D[(\partial^2 L/\partial\ddot q^i\,\partial\ddot q^j)\dddot q^j]\}$, which leads to terms containing the fifth-order derivatives of q^i and these must cancel each other irrespective of other terms (Bergmann 1949),

$$
b^{i\sigma}_{1(1)}\frac{\partial^2 L}{\partial \dot q^i_{(2)}\,\partial \dot q^j_{(2)}}\dddot q^j_{(2)}=0.
\tag{44}
$$

These conditions are to be fulfilled for any fifth-order derivative of q^i; one then obtains

$$
b^{i\sigma}_{1(1)}\frac{\partial^2 L}{\partial \dot q^i_{(2)}\,\partial \dot q^j}=0.
\tag{45}
$$

Because $b^{i\sigma}_{1(1)}$ are not all identically zero, which implies that the extended Hessian matrix (6) is degenerate and therefore this variant system has a Dirac constraint.

In the case of a system whose Lagrangian is gauge invariant or invariant up to an exact differential term under the transformation (40), we can proceed in the same way to conclude that the system also has a Dirac constraint.

6. The generators of gauge transformation

Gauge theories play an important role in modern field theories; these theories have a gauge invariance under the local transformation (or gauge transformation). Now we shall develop an algorithm to construct the generators of gauge transformation for a constrained Hamiltonian system with a singular second-order Lagrangian. For the sake of simplicity, all the constraints of the system are assumed to be first class. Under an infinitesimal gauge transformation suppose the two trajectories $(q^i_{(\alpha)}(t), p^{(\alpha)}_i(t), \lambda^q(t))$ and $(q^i_{(\alpha)}(t)+\delta q^i_{(\alpha)}(t), p^{(\alpha)}_i(t)+\delta p^{(\alpha)}_i(t), \lambda^q(t)+\delta\lambda^a(x))$ both satisfy the equations of motion (11) and constraint conditions (7), then the varied trajectory equations (11) and constraint conditions (7) can be expanded to first order in the small variations $\delta q^i_{(\alpha)}$, $\delta \dot{q}^i_{(\alpha)}$, $\delta p^{(\alpha)}_i$, $\delta \dot{p}^{(\alpha)}_i$ and $\delta\lambda^a$ and, using equations (11) and (7) for the unvaried trajectory, one finds

$$\delta \dot{q}^i_{(\alpha)} = \frac{\partial^2 H_T}{\partial q^j_{(\beta)}\, \partial p^{(\alpha)}_i}\, \delta q^j_{(\beta)} + \frac{\partial^2 H_T}{\partial p^{(\beta)}_j\, \partial p^{(\alpha)}_i}\, \delta p^{(\beta)}_j \qquad (\text{mod PC}) \tag{46a}$$

$$\delta \dot{p}^{(\alpha)}_i = -\left(\frac{\partial^2 H_T}{\partial q^j_{(\beta)}\, \partial q^i_{(\alpha)}}\, \delta q^j_{(\beta)} + \frac{\partial H_T}{\partial p^{(\beta)}_j\, \partial q^i_{(\alpha)}}\, \delta p^{(\beta)}_j\right) \qquad (\text{mod PC}) \tag{46b}$$

$$\frac{\partial \phi^0_a}{\partial q^i_{(\alpha)}}\, \delta q^i_{(\alpha)} + \frac{\partial \phi^0_a}{\partial p^{(\alpha)}_i}\, \delta p^{(\alpha)}_i = 0 \qquad (\text{mod PC}). \tag{46c}$$

Now let the variations of canonical variables be generated by a phase space function $G(q^i_{(\alpha)}, p^{(\alpha)}_i)$ and parametrized by an arbitrary infinitesimal function $\varepsilon(t)$, then, in general, one has to consider a generator of the type

$$G = \sum_{k=0}^{m} \varepsilon^{(k)} G_k = \sum_{k=0}^{m} (D^k \varepsilon) G_k \tag{47}$$

where the variation of $q^i_{(\alpha)}$ and $p^{(\alpha)}_i$ are given by

$$\delta q^i_{(\alpha)} = \sum_{k=0}^{m} \varepsilon^{(k)} \{q^i_{(\alpha)}, G_k\} = \sum_{k=0}^{m} \varepsilon^{(k)} \frac{\partial G_k}{\partial p^{(\alpha)}_i} \tag{48a}$$

$$\delta p^{(\alpha)}_i = \sum_{k=0}^{m} \varepsilon^{(k)} \{p^{(\alpha)}_i, G_k\} = -\sum_{k=0}^{m} \varepsilon^{(k)} \frac{\partial G_k}{\partial q^i_{(\alpha)}}. \tag{48b}$$

On substituting (48) into (46) one finds, owing to the arbitrariness of $\varepsilon(t)$, the following conditions on the G_k:

$$\frac{\partial}{\partial p^{(\alpha)}_i}(\{G_k, H_T\} + G_{k-1}) = 0 \qquad (\text{mod PC}) \tag{49a}$$

$$\frac{\partial}{\partial q^i_{(\alpha)}}(\{G_k, H_T\} + G_{k-1}) = 0 \qquad (\text{mod PC}) \tag{49b}$$

$$\{G_k, \phi^0_a\} = 0. \qquad (\text{mod PC}) \tag{49c}$$

Because we are considering variations that leave the trajectory on the constraint hypersurface one should add the further requirement that $\{G_k, \phi^n_a\} = 0$ to the third set of equations (49), the ϕ^n_a being all the secondary constraints that arise in the Dirac–Bergmann algorithm. Hence, all the G_k have to be FCC. H can be substituted instead of H_T, owing to the assumption that all the constraints are FCC. From (49) one finds the following recursive relations in a manner analogous to the discussion which was given by Castellani (1982):

$$\{G_0, H\} = 0 \qquad (\text{mod PC}) \tag{50a}$$

$$G_{k-1} + \{G_k, H\} = 0 \qquad (\text{mod PC}) \tag{50b}$$

$$G_m = 0. \qquad (\text{mod PC}) \tag{50c}$$

The generator G of the gauge transformation must be conservative; from expression (47) one gets

$$\dot{G} = \frac{\partial G}{\partial t} + \{G, H_T\} = \sum_{k=0}^{m} \varepsilon^{(k)} (G_{k-1} + \{G_k, H_T\}) = 0 \qquad (\text{mod PC}) \tag{51}$$

and owing to the arbitrariness of $\varepsilon(t)$ we conclude that the recursive relations in (50) are nothing but the conservation law of G.

Even when the SCC appear, if the series of the FCC derived from PC are completely separated from the series of second-class ones, this formulation on gauge symmetry is valid for such a system.

Therefore, we have generalized the Castellani results to the constrained Hamiltonian system with a singular second-order Lagrangian starting from another point of view. All the G_k have to be FCC and, with the exception of those FCC which arise as powers χ^n (Castellani 1982), are part of the gauge generators. The G_{k-1} is deduced from G_k according to the recursive relations (50*b*). Moreover, G_m must be a primary FCC for every primary FCC using (50) to construct the chains of G_k until G_0 is reached.

To illustrate the algorithm for the construction of the generator, we present an example. A model with the Lagrangian is given by

$$L=\dot{q}^1_{(2)}\dot{q}^2_{(2)}+q^1_{(2)}(q^2_{(2)}-q^3_{(2)})-q^1_{(1)}q^3_{(1)}. \tag{52}$$

The momenta conjugate to $q^i_{(\alpha)}$ are

$$p_1^{(2)}=\dot{q}^2_{(2)} \qquad p_2^{(2)}=\dot{q}^1_{(2)} \qquad p_3^{(2)}=0 \tag{53a}$$

$$p_1^{(1)}=q^2_{(2)}-q^3_{(2)}-\dot{p}_1^{(2)} \qquad p_2^{(1)}=q^1_{(2)}-\dot{p}_2^{(2)} \qquad p_3^{(1)}=-q^1_{(2)}. \tag{53b}$$

The PC is only

$$\phi^0=p_3^{(2)}=0. \tag{54}$$

The Hamiltonian is given by

$$H=\dot{q}^i_{(\alpha)}p_i^{(\alpha)}-L=p_1^{(2)}p_2^{(2)}+p_1^{(1)}q^1_{(2)}+p_2^{(1)}q^2_{(2)}+p_3^{(1)}q^3_{(2)}-q^1_{(2)}(q^2_{(2)}-q^3_{(2)})+q^1_{(1)}q^3_{(1)} \tag{55}$$

and the total Hamiltonian is given by

$$H_{\mathrm{T}}=H+\lambda\phi^0. \tag{56}$$

The stationary condition for constraints yield the following secondary constraints:

$$\phi^1=\{\phi^0, H_{\mathrm{T}}\}=-p_3^{(1)}-q^1_{(2)}=0 \tag{57a}$$

$$\phi^2=\{\phi^1, H_{\mathrm{T}}\}=q^1_{(1)}-p_2^{(2)}=0 \tag{57b}$$

$$\phi^3=\{\phi^2, H_{\mathrm{T}}\}=p_2^{(1)}=0. \tag{57c}$$

All the constraints $\phi^k (k=0, 1, 2, 3)$ are FCC. Let $G_3=\phi^0$, then, from recursive (50*b*), one finds $G_2=-\phi^1$, $G_1=\phi^2$, $G_0=-\phi^3$. According to expression (47) the generator of the gauge transformation is given by

$$G=-\varepsilon(t)\phi^3+\dot{\varepsilon}(t)\phi^2-\ddot{\varepsilon}(t)\phi^1+\dddot{\varepsilon}(t)\phi^0. \tag{58}$$

This generator produces the following transformation:

$$\begin{aligned}&\delta q^1_{(\alpha)}=\{q^1_{(\alpha)}, G\}=0 \qquad \delta q^2_{(\alpha)}=\{q^2_{(\alpha)}, G\}=-\varepsilon^{(\alpha)}(t)\\ &\delta q^3_{(\alpha)}=\{q^3_{(\alpha)}, G\}=\varepsilon^{(\alpha+2)}(t) \qquad \delta p_1^{(\alpha)}=\{p_1^{(\alpha)}, G\}=-\varepsilon^{(\alpha)}(t)\\ &\delta p_2^{(\alpha)}=\delta p_3^{(\alpha)}=0.\end{aligned} \tag{59}$$

Under this gauge transformation, one finds the invariance of the action

$$\delta L=-\frac{\mathrm{d}}{\mathrm{d}t}(\dot{q}^1\ddot{\varepsilon}+q^1\dot{\varepsilon}). \tag{60}$$

The action is invariant under the local transformation (59), hence there is a corresponding GNI (34) for this system. If all FCC in the expression (39) are taken into account this expression becomes the trivial identity. There is no contradiction with Dirac's conjecture for this example.

7. Application to the field theories

Consider the vector field with a scalar field whose Lagrangian density is given by

$$\mathscr{L}_0=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{c}{4}\partial_\sigma F_{\mu\nu}\partial^\sigma F^{\mu\nu}+\tfrac{1}{2}m^2B_\mu B^\mu-mB_\mu\partial^\mu\eta+\tfrac{1}{2}\partial_\mu\eta\partial^\mu\eta \tag{61}$$

where the field strength tensor is expressed in terms of potentials in the usual way, $F_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu$, and c is a constant. The momenta $\pi(x)$ conjugate to the scalar field $\eta(x)$ is

$$\pi=\frac{\delta L_0}{\delta(\partial_0\eta)}=-mB^0+\dot{\eta}. \tag{62}$$

The momenta $\pi_\mu^{(1)}$ and $\pi_\mu^{(2)}$ conjugate to the vector fields $B^\mu_{(1)}\equiv B^\mu$ and $B^\mu_{(2)}\equiv\dot{B}^\mu$ are

$$\pi_0^{(2)}=0 \tag{63a}$$

$$\pi_i^{(2)}=c\partial_0F_{0i} \tag{63b}$$

$$\pi_0^{(1)}=c\partial_0\,\partial^iF_{0i} \tag{63c}$$

$$\pi_i^{(1)}=c(\nabla^2F_{0i}+\partial_0\,\partial^jF_{ji})+F_{0i}-\partial_0\pi_i^{(2)} \tag{63d}$$

respectively. The Hamiltonian is given by

$$\begin{aligned}H_0&=\int \mathrm{d}^3x\,\mathscr{H}_0\\&=\int \mathrm{d}^3x\bigg(\frac{1}{2c}(\pi_i^{(2)})^2+\tfrac{1}{2}\pi^2+\pi_i^{(2)}\partial^iB^0_{(2)}+\frac{c}{4}\partial_0F_{ij}\partial^0F^{ij}+\frac{c}{2}\partial_iF_{0j}\partial^iF^{0j}\\&\quad+\frac{c}{4}\partial_iF_{jk}\partial^iF^{jk}+\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\pi_\mu^{(1)}B^\mu_{(2)}+\tfrac{1}{2}m^2B_iB^i+\tfrac{1}{2}\nabla\eta\cdot\nabla\eta-mB_i\partial^i\eta\\&\quad-(\partial^i\pi_i^{(1)}+m\pi)B^0\bigg).\end{aligned} \tag{64}$$

The PC is only

$$\phi^0=\pi_0^{(2)}. \tag{65}$$

The total Hamiltonian is given by

$$H_\mathrm{T}=\int \mathrm{d}^3x(\mathscr{H}_0+\lambda\phi^0). \tag{66}$$

The stationarity for the constraints yields the following secondary constraints:

$$\phi^1=\{\phi^0,H_\mathrm{T}\}=-\pi_0^{(1)}+\partial^i\pi_i^{(2)}=0 \tag{67}$$

$$\phi^2=\{\phi^1,H_\mathrm{T}\}=\partial^i\pi_i^{(1)}+m\pi=0 \tag{68}$$

where the Poisson bracket for canonical variables $(\psi^{\pi}_{(\alpha)}, \pi^{(\alpha)}_{n})$ is given by

$$\{U, V\} = \int d^3x \left(\frac{\delta U}{\delta \psi^{n}_{(\alpha)}} \frac{\delta V}{\delta \pi^{(\alpha)}_{n}} - \frac{\delta U}{\delta \pi^{(\alpha)}_{n}} \frac{\delta V}{\delta \psi^{\pi}_{(\alpha)}} \right). \tag{69}$$

All the constraints $\phi^k (k=0,1,2)$ are FCC, and according to (47) and (50) the generator of the gauge transformation is

$$G = \int d^3x (\pi^{(1)}_{\mu} \partial^{\mu} \varepsilon + m\pi\varepsilon + \pi^{(2)}_{\mu} \partial_0 \partial^{\mu} \varepsilon). \tag{70}$$

The gauge transformations induced by G are

$$\delta B^{\mu}_{(1)} = \{B^{\mu}_{(1)}, G\} = \partial^{\mu}\varepsilon \qquad \delta B^{\mu}_{(2)} = \partial_0 \partial^{\mu} \varepsilon \qquad \delta\eta = m\varepsilon \tag{71a}$$

$$\delta \pi^{(1)}_{\mu} = \delta \pi^{(2)}_{\mu} = \delta \pi = 0. \tag{71b}$$

Under the transformation (71), the Lagrangian is gauge invariant.

Let us consider the vector field B^{μ} coupling with the external source $j_{\mu} = (\rho, j)$; the Lagrangian density is then given by

$$\mathscr{L} = \mathscr{L}_0 + B^{\mu} j_{\mu}. \tag{72}$$

Under the transformation (71), this Lagrangian is not invariant, and in this case the GNI (34) becomes

$$\partial^{\mu}\left(\dot{\pi}^{(1)}_{\mu} + \frac{\delta H}{\delta B^{\mu}_{(1)}} \right) - \partial_0 \partial^{\mu} \left(\dot{\pi}^{(2)}_{\mu} + \frac{\delta H}{\delta B^{\mu}_{(2)}} \right) - m \left(\dot{\pi} + \frac{\delta H}{\delta \eta} \right) = -\partial^{\mu} j_{\mu} \tag{73}$$

where $H = \int d^3x (\mathscr{H}_0 - B^{\mu} j_{\mu})$. The total Hamiltonian of this system is given by

$$H_T = \int d^3x (\mathscr{H}_0 - B^{\mu} j_{\mu} + \lambda \phi^0). \tag{74}$$

According to the equations of motion (9) of this constrained Hamiltonian system, from (65), (73) and (74), along the trajectory of motion one obtains

$$\partial^{\mu} j_{\mu} = 0. \tag{75}$$

If the equations of motion are derived from the extended Hamiltonian $H_E = H_T + \int d^3x [\mu_1 \phi^1 + \mu_2 (\phi^2 - \rho)]$, where ϕ^1 and ϕ^2 are given by (67) and (68) respectively. One can obtain the same result (75). Therefore this conservation law valid whether Dirac's conjecture holds true or not.

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